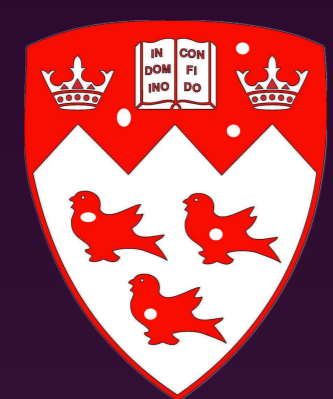
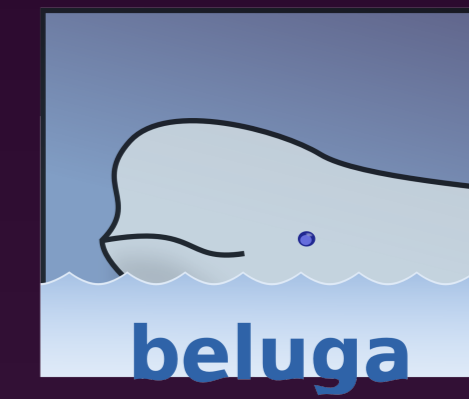


Well-founded Recursion over Contextual Objects



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Introduction

Today, we routinely specify and reason about the runtime behavior of software using formal systems such as type systems to establish safety properties. The Logical framework LF is a powerful system for encoding formal systems, but justifying well-founded recursion principles over LF specifications remains a long-standing problem in the area of mechanizing proofs. **Our current contribution is the design of a well-founded induction principle over LF terms and a consistency proof based on logical relations.**

Features of our language

We concentrate our work on **Beluga** [Pientka, 2008], a novel language with supports for specifying formal systems in LF and reasoning with contextual objects, i.e. a LF object paired with a context. Our language is a simply-typed Beluga combining **contextual types** [Nanevski et al., 2008] with a **well-founded primitive recursion principle**.

► Represent abstract syntax of the object language

```
datatype term: type =
| lam: (term → term) → term
| app: term → term → term;
schema ctx = term;
```

- The object language is a simply-typed fragment λ^{\rightarrow}
- Model binders in the object language by binders in the meta-language
- Assume the basic computation-level type (int)

Motivating Examples:

• Formalize the object language in LF

• Count constructors

Object language \rightsquigarrow **Meta-language (LF)**

```
lam x.x  $\rightsquigarrow$  lam ( $\lambda x.x$ )
lam x.lam y.x y  $\rightsquigarrow$  lam ( $\lambda x.lam (\lambda y.app x y)$ )
```

cntcons (lam ($\lambda x.x$)) = 1

cntcons (lam ($\lambda x.lam (\lambda y.app x y)$)) = 3

► Traverse λ -terms and analyze open data

```
lam ( $\lambda x.lam (\lambda y. \text{I am a hole. My type is } [x,y.term]. \text{ )))$ 
```

- Characterize a hole as a contextual object $[g.M]$ (M can depend on g)
- The local context g may grow when traversing terms

► Express the counting function in Beluga-like syntax

```
rec cntcons: (g:ctx) [g.term] → int =
fn y ⇒ case y of
| [g.#p.] ⇒ 0 % base case
| [g.lam  $\lambda x.M..x$ ] ⇒ 1 + cntcons [g,x:term.M..x] % step case
| [g.app (M..) (N..)] ⇒ 1 + cntcons [g.M..] + cntcons [g.N..]; % step case
```

► Formal representation of the counting function

- **Challenge:** To generate the appropriate recursive calls
- Solution:** $f [h,x:term] [h,x.M..x]$ denotes the valid recursive calls
- **Challenge:** To extend the context when appealing to recursive calls
- Solution:** Introduce context variable h : ctx in each branch

```
cntcst =  $\Lambda g. \lambda y. rec^{[g.term] \leftarrow int} (y,$ 
  h: ctx, #p: h.term.
    f [h] [h.#p.]  $\Rightarrow$  0
  h: ctx, M: h,x:term.term. f [h,x:term] [h,x.M..x]: int.
    f [h] [h.lam  $\lambda x.M..x$ ]  $\Rightarrow$  1 + f [h,x:term] [h,x.M..x]
  h: ctx, M: h.term, N: h.term. f [h] [h.M..]: int, f [h] [h.N..]: int.
    f [h] [h.app (M..) (N..)]  $\Rightarrow$  1 + f [h] [h.M..] + f [h] [h.N..]
```

Programs as proofs

► Generalize recursive functions to encode inductive proofs

Proofs depending on assumptions are characterized by contextual objects

Inductive proofs \rightsquigarrow Recursive programs
Apply induction hypotheses \rightsquigarrow Make recursive calls
Cases in a proof \rightsquigarrow Branches in a function

► To justify well-founded recursive functions implement well-founded inductive proofs

- Branches must be complete \Leftarrow Verified by the **splitting & coverage algorithm**
- Programs must terminate \Leftarrow Verified by the **weak normalization**

These two verifications guarantee the **consistency** of our language.

Splitting & coverage checking algorithm

► Splitting & coverage checking algorithm

$$\text{split } (g \vdash g.P) = \frac{}{\{ (g, \Delta \vdash M: g.P, f [h] [h.N]) \}}$$

Incorporate the algorithm into the type system to perform the coverage checking

► Example: splitting on a contextual object of type *term*

$$\text{split } (g \vdash g.term) = \{ (g, \#p: g.term \vdash g.\#p.: g.term), (g, M: g,x:term.term \vdash g.lam \lambda x.M..x: g.term, f [g,x:term] [g,x.M..x]), (g, M: g.term, N: g.term \vdash g.app (M..) (N.): g.term, f [g] [g.M..], f [g] [g.N..]) \}$$

► Properties of the algorithm

- We proved the algorithm generates all valid primitive recursive calls.
- We proved the soundness of the algorithm.

Weak Normalization

► Understanding what the weak normalization is about

Weak Normalization Theorem

If a closed term e is well-typed (i.e. $\cdot \vdash e : \tau$), then there exists a finite reduction sequence of e (i.e. e is weakly normalizing).

The weak normalization and the determinacy of the evaluation verify the termination.

► Principal idea underlying our proof of the weak normalization

We build the proof by proving the following lemmas together with some auxiliary lemmas, following the classical *logical relations* technique invented by Tait [1967].

If $\cdot \vdash e : \tau$, then $e \in \mathcal{R}_\tau$.
(e has type τ infers e is reducible at τ)

If $e \in \mathcal{R}_\tau$, then e is weakly normalizing.
(e is reducible infers e is weakly normalizing)

► Defining Reducibility Candidate \mathcal{R}_τ

We define \mathcal{R}_τ inductively on type τ , building on the work from Lindley and Stark [2005].

Example: $e \in \mathcal{R}_{[g.P]}$ if (1) e is weakly normalizing (2) $\cdot \vdash e : [g.P]$
define $\mathcal{R}_{[g.P]}$ (3) for all $\vec{b} \in \mathcal{R}_{\Pi h.[h.P] \leftarrow \tau}$, $\text{rec}(e, \vec{b})$ is weakly normalizing

Conclusion

Our main contributions towards justifying well-founded induction principle over LF terms:

- Defining a call-by-value small-step semantics
- Designing a sound splitting and coverage checking algorithm
- Proving the weak normalization (the consistency of the simply-typed Beluga)

Future work

- Extending the simply-typed framework to the dependently-typed setting
- Progressing from primitive recursive functions to general recursive functions
- Developing a well-founded recursion principle for context variables

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Acknowledgements

I want to express my sincere gratitude to my supervisor, Professor Brigitte Pientka, for her patient supervision, expert guidance, and continued encouragement.